



TITLE:

A duality for finite t-modules

AUTHOR(S):

Taguchi, Yuichiro

CITATION:

Taguchi, Yuichiro. A duality for finite t-modules. 数理解析研究所講究録
1992, 797: 127-131

ISSUE DATE:

1992-08

URL:

<http://hdl.handle.net/2433/82774>

RIGHT:

A duality for finite t -modules

都立大理 田口 雄一郎 (Yuichiro Taguchi)

The duality mentioned in the title is the $\mathbb{F}_q[t]$ -analogue of the Cartier duality, replacing the multiplicative group G_m by the Carlitz module C . Finite t -modules are, roughly, finite locally free group schemes which are $\mathbb{F}_q[t]$ -submodules of abelian t -modules ([1]) with scalar t -action on their tangent spaces. The duality is expected to play a fundamental role in the theory of Drinfeld motives. Throughout the article, \mathcal{O}_S denotes the structure sheaf of a scheme S .

1. Definitions

Let A be any commutative ring. For an A -scheme S , we denote by $\alpha : A \rightarrow \Gamma(S, \mathcal{O}_S)$ the structure morphism.

DEFINITION. An A -module scheme over an A -scheme S is a pair (G, Ψ) consisting of a commutative group scheme G over S and a ring homomorphism $\Psi : A \rightarrow \text{End}(G/S)$; $a \mapsto \Psi_a$ such that, for each $a \in A$, Ψ_a induces multiplication by $\alpha(a)$ on the \mathcal{O}_S -module $\text{Lie}(G/S)$.

EXAMPLE. A vector bundle G on S can be naturally regarded as a $\Gamma(S, \mathcal{O}_S)$ -module scheme. We shall mean by a *vector group scheme* such a $\Gamma(S, \mathcal{O}_S)$ -module scheme.

Now we define several “modules” and “sheaves” (M_i and S_i for $i = 1, 2, 3$ below). The morphisms are defined naturally for them (though we omit the definition). Each “modules” and “sheaves”, for $i = 1, 2, 3$, are in anti-equivalence of categories.

First, let $A = \mathbb{F}_q$, and let S be an \mathbb{F}_q -scheme. For an \mathbb{F}_q -module scheme (G, Ψ) over S , set $\mathcal{E}_G := \underline{\text{Hom}}_{\mathbb{F}_q, S}(G, G_a)$. ($\underline{\text{Hom}}_{\mathbb{F}_q, S}$ denotes the Zariski sheaf on S of \mathbb{F}_q -linear homomorphisms.)

DEFINITION M1. An \mathbb{F}_q -module scheme (G, Ψ) over S is called a *finite φ -module* if \mathcal{O}_G and \mathcal{E}_G are locally free of finite rank over \mathcal{O}_S with $\text{rank } \mathcal{O}_G = q^{\text{rank } \mathcal{E}_G}$, and \mathcal{E}_G generates the \mathcal{O}_S -algebra \mathcal{O}_G .

DEFINITION S1. (Drinfeld [2], §2) A φ -sheaf is a pair (\mathcal{E}, φ) consisting of a locally free \mathcal{O}_S -module \mathcal{E} on S of finite rank and an \mathcal{O}_S -module homomorphism $\varphi : \mathcal{E}^{(q)} \rightarrow \mathcal{E}$. (Here $\mathcal{E}^{(q)}$ denotes the base extension of \mathcal{E} by the q -th power map $\mathcal{O}_S \rightarrow \mathcal{O}_S$.)

In the rest, A is the polynomial ring $\mathbb{F}_q[t]$ in one variable t over \mathbb{F}_q , and S is an A -scheme. Set $\theta := \alpha(t)$, the image of t in \mathcal{O}_S .

DEFINITION M2. A *finite t -module* (G, Ψ) over S is an A -module scheme over S such that

- (1) G is killed by some $a \in A - \mathbb{F}_q$; and
- (2) $(G, \Psi|_{\mathbb{F}_q})$ is a finite φ -module over S .

DEFINITION S2. A *t -sheaf* $(\mathcal{E}, \varphi, \psi_t)$ on S is the pair of a φ -sheaf (\mathcal{E}, φ) and an endomorphism ψ_t of (\mathcal{E}, φ) which induces multiplication by θ on $\text{Coker}(\varphi)$. (Recall that $\text{Coker}(\varphi)$ is canonically isomorphic to $\text{Lie}^*\text{Gr}(\mathcal{E}, \varphi)$ ([2], Proposition 2.1, 2)).)

We note here that a finite φ -module G can be canonically embedded into a vector group scheme $E_G := \underline{\text{Spec}}(\text{Sym}_{\mathcal{O}_S} \mathcal{E}_G)$.

DEFINITION M3. A *finite v -module* (G, Ψ, V) over S is a finite t -module scheme (G, Ψ) over S together with a morphism $V : E_G^{(q)} \rightarrow E_G$ of \mathbb{F}_q -module schemes such that $\Psi_t = (\theta + V \circ F_{E_G})|_G$. (Here θ means multiplication by $\theta = \alpha(t) \in \Gamma(S, \mathcal{O}_S)$ on E_G , and F_{E_G} is the Frobenius morphism of E_G .)

DEFINITION S3. A *v -sheaf* $(\mathcal{E}, \varphi, v)$ on S is the pair of a φ -sheaf (\mathcal{E}, φ) on S and an \mathcal{O}_S -module homomorphism $v : \mathcal{E} \rightarrow \mathcal{E}^{(q)}$ such that $(\mathcal{E}, \varphi, \psi_t)$ with $\psi_t := \theta + \varphi \circ v$ is a t -sheaf on S . (Here θ means multiplication by θ on \mathcal{E} .)

EXAMPLE. Let (E, Ψ) be a Drinfeld A -module of rank r over $S = \text{Spec } R$, where R is an A -algebra. Assume the action of t is given by

$$\psi_t(X) = \theta X + a_1 X^q + \cdots + a_r X^{q^r}, \quad a_i \in R, \quad a_r \in R^\times,$$

with respect to a trivialization $E \simeq \mathbb{G}_a = \text{Spec } R[X]$. Then for $a \in A - 0$, the finite t -module $G = \text{Ker}(\Psi_a)$ is furnished with a v -module structure by

$$\begin{aligned} v : \mathcal{E}_G &\rightarrow \mathcal{E}_G^{(q)}, \\ X^{q^i} &\mapsto X^{q^{i-1}} \otimes (\theta^{q^i} - \theta) + X^{q^i} \otimes a_1^{q^i} + \cdots + X^{q^{r+i-1}} \otimes a_r^{q^i}. \end{aligned}$$

(Here $X^{q^{i-1}} \otimes (\theta^{q^i} - \theta) := 0$ if $i = 0$.) But this v -module structure is not unique unless the Frobenius morphism is injective on $\mathcal{E}_G^{(q)}$.

In fact, finite v -modules over “mixed characteristic” bases are not so far from finite t -modules, since we have:

PROPOSITION. Let (G, Ψ) be a finite t -module which is étale over the generic points of S . Then (G, Ψ) has a unique v -module structure V_G extending the given t -module structure; $\Psi_t = (\theta + V_G \circ F_{E_G})|_G$. If G and G' are two such finite t -modules, then a morphism $G \rightarrow G'$ of finite t -modules preserves this v -module structure. In particular, if $\alpha : A \rightarrow \mathcal{O}_S$ is injective, then the two concepts, a finite t -module and a finite v -module, are equivalent.

The same is valid for a t -sheaf $(\mathcal{E}, \varphi, \psi_t)$ such that $\varphi : \mathcal{E}^{(q)} \rightarrow \mathcal{E}$ is injective.

2. The duality

For an \mathcal{O}_S -module \mathcal{E} , put $\mathcal{E}^* := \underline{\text{Hom}}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_S)$. If $(\mathcal{E}, \varphi, v)$ is a v -sheaf on S , then φ and v induce respectively the \mathcal{O}_S -module homomorphisms

$$\varphi^* : \mathcal{E}^* \rightarrow \mathcal{E}^{*(q)} \quad \text{and} \quad v^* : \mathcal{E}^{*(q)} \rightarrow \mathcal{E}^*.$$

It is easy to check that $(\mathcal{E}^*, v^*, \varphi^*)$ is a v -sheaf on S .

DEFINITION. We define the *dual* $(\mathcal{E}, \varphi, v)^*$ of a v -sheaf $(\mathcal{E}, \varphi, v)$ to be the v -sheaf $(\mathcal{E}^*, v^*, \varphi^*)$. If a finite v -module G corresponds to a v -sheaf $(\mathcal{E}, \varphi, v)$, then we define its *dual* G^* to be the finite v -module which corresponds to the v -sheaf $(\mathcal{E}^*, v^*, \varphi^*)$.

We have clearly the following

PROPOSITION. *Let G be a finite v -module.*

- (i) G^* has the same rank as G .
- (ii) The correspondence $G \mapsto G^*$ is functorial. This functor is exact.
- (iii) G^{**} is canonically isomorphic to G .
- (iv) $(G \times_S T)^* \simeq G^* \times_S T$ for any S -scheme T .

The same is true for the duality of v -sheaves.

Our main result is the following

THEOREM. *Let C be the Carlitz module over $\text{Spec } A$, and let G be a finite v -module over S .*

- (i) *The functor*

$$\begin{aligned} \underline{\text{Hom}}_{v,S} : (S\text{-schemes}) &\rightarrow (A\text{-modules}) \\ T &\mapsto \text{Hom}_{v,T}(G \times_S T, C \times_{\text{Spec } A} T) \end{aligned}$$

is represented by (the underlying finite t -module of) G^ .*

- (ii) *There exists an A -bilinear pairing of A -module schemes:*

$$\Pi : G \times_S G^* \rightarrow C$$

such that:

- (ii-1) *If G' is a finite t -module over S sitting in an A -bilinear pairing $\Pi' : G \times_S G' \rightarrow C$, then there exists a unique morphism $M : G' \rightarrow G^*$ of finite t -modules which makes the diagram*

$$\begin{array}{ccc} G \times_S G' & \xrightarrow{\Pi'} & C \\ 1 \times M \downarrow & & \parallel \\ G \times_S G^* & \xrightarrow{\Pi} & C \end{array}$$

commute.

(ii-2) If $\alpha : A \rightarrow \mathcal{O}_S$ is injective and S is integral with function field K , then Π induces a non-degenerate A -bilinear pairing between the A -modules of geometric points:

$$G(K^{\text{sep}}) \times G^*(K^{\text{sep}}) \rightarrow C(K^{\text{sep}}).$$

If we consider only the t -module structure, we will have the following:

(i) The functor

$$\begin{aligned} \underline{\text{Hom}}_{t,S}(G, C) : (S\text{-schemes}) &\rightarrow (A\text{-modules}) \\ T &\mapsto \text{Hom}_{t,T}(G \times_S T, C \times_{\text{Spec } A} T) \end{aligned}$$

is represented by an A -module scheme \tilde{G}^* over S .

(ii) If G is étale over the generic points of S , then \tilde{G}^* is of the form $G^* \cup \tilde{G}_0^*$, where G^* is (the underlying finite t -module of) the dual finite v -module of G with the unique v -module structure, and \tilde{G}_0^* is supported on the locus in S where G is not étale. In general, \tilde{G}_0^* has a positive dimension.

Finally, we mention the Frobenius-Verschiebung relation over a “finite characteristic” base.

PROPOSITION. Let (G, Ψ, V) be a finite v -module over S .

(i) Let d be a positive integer, and $F_G^d : G \rightarrow G^{(q^d)}$ the q^d -th power Frobenius morphism. Then $G^{(q^d)}$ (resp. F_G^d) is a finite v -module (resp. a morphism of finite v -modules) if and only if $\text{Im}(\alpha) \subset \mathbb{F}_{q^d}$.

(ii) Assume $\text{Ker}(\alpha : A \rightarrow \mathcal{O}_S) = (\mathfrak{p})$ with $\mathfrak{p} \in A$ being a monic prime element of degree d . Let $V_{G,\mathfrak{p}} : G^{(q^d)} \rightarrow G$ be the dual morphism of $F_{G^*,\mathfrak{p}} := F_{G^*}^d : G^* \rightarrow G^{*(q^d)}$. Then we have

$$\Psi_{\mathfrak{p}} = V_{G,\mathfrak{p}} \circ F_{G,\mathfrak{p}} \quad \text{and} \quad \Psi_{\mathfrak{p}}^{(q^d)} = F_{G,\mathfrak{p}} \circ V_{G,\mathfrak{p}}.$$

In particular, we have an exact sequence of finite t -modules

$$0 \rightarrow \text{Ker}(F_{G,\mathfrak{p}}) \rightarrow \text{Ker}(\Psi_{\mathfrak{p}}) \rightarrow \text{Ker}(V_{G,\mathfrak{p}}) \rightarrow 0.$$

3. Comments

(1) Our theory is almost the “Dieudonné theory” for finite t -modules. It might be possible to develop the theory of universal extensions of abelian t -modules from our point of view.

(2) The relation between our duality and the dual isogeny of abelian t -modules remains to be worked out.

(3) I have not considered the case of general A , the ring of elements of a function field over a finite field which are regular outside a fixed place.

(4) Does there exist such a duality for “finite π -modules”, with target in arbitrary Lubin-Tate group? To what extent are the coefficients $\mathbb{Q}_p(r)$ celestial (or godgiven)? That is, in which cases can one replace the coefficients $\mathbb{Q}_p(r)$ by other local fields with other “Lubin-Tate twists”?

(5) Our duality will be used to prove the “Carlitz-Hodge-Tate decomposition”, which will be reported elsewhere.

References

- [1] G. W. Anderson, t -motives, Duke math. J. 53 (1986), 457–502
- [2] V. G. Drinfeld, Moduli variety of F -sheaves, Funktsional’nyi Analiz i Ego Prilozheniya 21 (1987), 23– 41

Tokyo Metropolitan University, Hachioji, Tokyo, 192-03 JAPAN

taguchi@math.metro-u.ac.jp